On classification of locally finite simple groups

Yaroslav Lavrenyuk

National Taras Shevchenko University of Kyiv

Ljubljana. June 6, 2007

Abstract

This is a joint work[1] with V.Nekrashevych. We classify locally finite groups, which are inductive limits of direct products of alternating groups with respect to block-diagonal embeddings. This class of groups includes a well known class of simple locally finite groups (so-called LDA-groups). We show that two such groups are isomorphic if and only if the AF-algebras defined by the respective Bratteli diagrams are isomorphic. Then the classical results on classification of AF-algebras can be applied.

[1] Yaroslav Lavrenyuk and Volodymyr Nekrashevych *On classification of inductive limits of direct products of alternating groups*, Journal of the London Mathematical Society, **75**, (2007), No. 1, p. 146-162.

A group is called *locally finite* group if every finite set of elements of the group generates finite subgroup.
Obviously that a direct limit of simple finite groups is a simple locally finite group. But not every simple locally finite group can be represented as a direct product of simple finite groups.

Classification of locally finite simple groups is very far from being complete, but several classes of such groups were studied in detail. It is known[2] that a locally finite simple group which has a faithful representation as a linear group in finite dimension over a field is isomorphic to a Lie type group over a locally finite filed (that is, an infinite sub-field of $\overline{\mathbb{F}_p}$, for some prime p).

---

[2]V. V. Belyaev, *Locally finite Chevalley groups*, Studies in Group Theory, Akad. Nauk SSSR, Ural Scientific Center, Sverdlovsk, 1984, pp. 39–88.
A. V. Borovik, *Periodic linear groups of odd characteristic*, Soviet Math. Dokl. **26** (1982), 484–486.
J.I. Hall and B. Hartley, *A group theoretical characterisation of simple, locally finite, finitary linear groups*, Arch. Math. **60** (1993), no. 2, 108–114.
S. Thomas, *The classification of simple periodic linear groups*, Arch. Math. **41** (1983), 103–116.

Next well studied class are the *finitary groups*. A *finitary linear representation* of a group G is a linear representation ρ such that for every $g \in G$ the kernel $\ker(1 - \rho(g))$ has finite co-dimension. A group is called *finitary linear* if it has a faithful finitary linear representation.

The list of possible finitary linear groups[3] is

1. an infinite alternating group Alt_Ω for infinite Ω;
2. a finitary symplectic group $FSp_K(V, s)$;
3. a finitary special unitary group $FSU_K(V, u)$;
4. a finitary orthogonal group $F\Omega_K(V, q)$;
5. a special transvection group $T_K(W, V)$.

Here K is a (possibly finite) subfield of $\overline{\mathbb{F}_p}$, for some prime p; the forms s, u, and q are nondegenerate on the infinite dimensional K-space V; and W is a subspace of the dual V^* whose annihilator in V is trivial.

[3] J. I. Hall, *Locally finite simple groups of finitary linear transformations*, Finite and locally finite groups (Istanbul, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 471, Kluwer Acad. Publ., Dordrecht, 1995, pp. 147–188.

The non-finitary locally finite simple groups are understood much worse. There exists a rough division of this class into groups of "1-type, or ∞-type, or p-type". This division uses the notion of a Kegel cover, defined in the following way.

Definition

A set of pairs $\{(H_i, M_i) \mid i \in I\}$ is called a *Kegel cover* for a locally finite group G if, for all $i \in I$, H_i is a finite subgroup of G and M_i is a maximal normal subgroup of H_i, and for each finite subgroup H of G there exists $i \in I$ with $H \leq H_i$ and $H \cap M_i = 1$. The groups H_i/M_i, $i \in I$, are called the *factors* of the Kegel cover.

Every simple locally finite group has a Kegel cover. Using Kegel covers many questions about locally finite simple groups can be transfered to questions about finite simple groups.

Definitions of classes of non-finitary simple locally finite groups

- A non-finitary locally finite simple group is said to be of 1-*type* if every Kegel cover of G has a factor which is an alternating group.
- If p is a prime, then G is of p-*type* if G is non-finitary and every Kegel cover of G has a factor which is isomorphic to a classical group defined over a field of characteristic p.
- A non-finitary locally finite simple group is said to be of ∞-*type* if for any class $\mathcal{G}$ of finite simple groups, such that every finite group can be embedded into a member of $\mathcal{G}$, there exists a Kegel cover of G all of whose factors are isomorphic to a member of $\mathcal{G}$.

U. Meierfrankenfeld and S. Delcroix proved[4] that if G is a locally finite simple group then exactly one of the following possibilities holds:

1. G is finitary;
2. G is of 1-type;
3. G is of p-type for a unique p;
4. G is of ∞-type.

---

[4] U. Meierfrankenfeld, *Non-finitary locally finite simple groups*, Finite and Locally Finite Groups (B. Hartley et al., ed.), Kluwer Academic, Dordrecht, 1995, pp. 189–212.
Stefaan Delcroix and Ulrich Meierfrankenfeld, *Locally finite simple groups of 1-type*, J. Algebra **247** (2002), no. 2, 728–746.

An important class of groups of 1-type are the inductive limits of direct products of alternating groups with respect to block-diagonal embeddings (also called *LDA*-groups[5]). The class of such groups does not exhaust the class of groups of 1-type, but the structure of an arbitrary group of 1-type seems to be similar to the structure of an *LDA*-group[6].

Definition

LDA-groups are direct limits of direct products $H_i = A_{i1} \times \ldots \times A_{ir_i}$ ($i \in \mathbb{N}$) of finite alternating groups $A_{ik} = \mathrm{Alt}\,(X_{ik})$ such that, for all $i < j$, every non-trivial orbit of any A_{ik} on any X_{jl} is natural.

The LDA-groups can be defined in the terms of Bratteli diagrams.

[5] Felix Leinen and Orazio Puglisi, *Some results concerning simple locally finite groups of 1-type*, Journal of Algebra **287** (2005), 32–51.

[6] Stefaan Delcroix and Ulrich Meierfrankenfeld, *Locally finite simple groups of 1-type*, J. Algebra **247** (2002), no. 2, 728–746.

A Bratteli diagram $\mathsf{B} = (\{V_i\}_{i\geq 0}, \{E_i\}_{i\geq 1}, \mathsf{s}, \mathsf{r}, d)$ is defined by the following data

- the set of *vertices* $V = V(\mathsf{B})$ together with a partition into a disjoint union $V = \bigsqcup_{i\geq 0} V_i$ of *levels*;
- the set of *edges*, or *arrows* $E = E(\mathsf{B})$ also partitioned into a disjoint union $E = \bigsqcup_{i\geq 1} E_i$;
- *source* and *range* maps $\mathsf{s} : E_i \longrightarrow V_{i-1}$ and $\mathsf{r} : E_i \longrightarrow V_i$, if $e \in E$ is an arrow, then $\mathsf{s}(e)$ is its *beginning* and $\mathsf{r}(e)$ is its *end*;
- a *labeling* $d : V \longrightarrow \mathbb{N} \setminus \{0\}$ of vertices by positive integers

We require that every vertex is a starting vertex of an edge and that

$$d(v) \geq \sum_{\mathsf{r}(e)=v} d(\mathsf{s}(e)). \qquad (1)$$

Note that we allow more than one edge e starting and ending in the same pair of vertices $\mathsf{s}(e), \mathsf{r}(e)$ (and then the corresponding summand will appear several times in (1)).

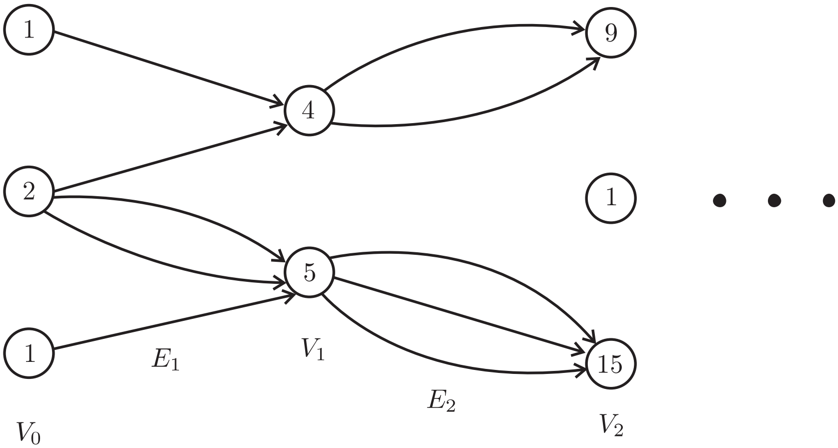

Figure: A Bratteli diagram

Definition

Let B be a Bratteli diagram and let A_i and $\delta : A_i \hookrightarrow A_{i+1}$ be the direct products of alternating groups $\prod_{v \in V_i} \mathrm{Alt}_{d(v)}$ and the embeddings defined by the Bratteli diagram B in natural way. Then the direct limit $\lim_{\rightarrow} A_i$ is denoted $A(\mathrm{B})$ and is called the *alternating group* of the Bratteli diagram.

Observation: The class of alternating groups of Bratteli diagrams coincides with the class of LDA-groups.

Dimension group of a Bratteli diagram

We will not define the K-theory of C^*-algebras, but rather describe how to construct the dimension group (the K_0-group with additional structure) of the AF-algebra $\mathfrak{A}(\mathrm{B})$ directly from the Bratteli diagram B. This group will be called the *dimension group of the Bratteli diagram* B and will be denoted $\Delta(\mathrm{B})$.

Let $\mathsf{B} = (\{V_i\}, \{E_i\}, \mathsf{s}, \mathsf{r}, d)$ be a Bratteli diagram. Consider the free abelian group

$$\mathbb{Z}^{|V_i|} = \{(n_v)_{v \in V_i} \ : \ n_v \in \mathbb{Z}\}$$

whose coordinates are labeled by the vertices of the ith level, together with the *positive cone*

$$\mathbb{Z}_+^{|V_i|} = \{(n_v)_{v \in V_i} \ : \ n_v \geq 0\}$$

and the *scale*

$$\Gamma_i = \{(n_v)_{v \in V_i} \ : \ 0 \leq n_v \leq d(v)\}.$$

The triple $\Delta_i = \left(\mathbb{Z}^{|V_i|}, \mathbb{Z}_+^{|V_i|}, \Gamma_i\right)$ is called the *dimension group* of the level number i.

Let us denote by ε_v, for every $v \in V_i$, the basis vector of $\mathbb{Z}^{|V_i|}$ corresponding to v, i.e., the vector $(n_u)_{u \in V_i}$, where $n_v = 1$ and $n_u = 0$ for all $u \neq v$.
Then the Bratteli diagram defines naturally homomorphisms $D : \mathbb{Z}^{|V_i|} \longrightarrow \mathbb{Z}^{|V_{i+1}|}$ mapping the basis vector ε_v to

$$D(\varepsilon_v) = \left(\sum_{e \in E_{i+1}, \mathsf{s}(e) = v} \varepsilon_{\mathsf{r}(e)} \right) \in \mathbb{Z}^{|V_{i+1}|}.$$

In other terms, D is the linear map defined by the *adjacency matrix* of the ith level of the Bratteli diagram.
It is easy to see that $D\left(\mathbb{Z}_+^{|V_i|}\right) \subseteq \mathbb{Z}_+^{|V_{i+1}|}$. We also have $D(\Gamma_i) \subseteq \Gamma_{i+1}$ due to (1).

Definition

The *dimension group* of the Bratteli diagram is the triple $\Delta(\mathrm{B}) = \big(K_0(\mathrm{B}), K_0^+(\mathrm{B}), \Gamma(\mathrm{B})\big)$, where $K_0(\mathrm{B})$ is the direct limit of the groups $\mathbb{Z}^{|V_i|}$ with respect to the embeddings D, $K_0^+(\mathrm{B})$ (the *positive cone*) is the union of the sets $\mathbb{Z}_+^{|V_i|}$ and $\Gamma(\mathrm{B})$ (the *scale*) is the union of the sets Γ_i in the inductive limit.
Two dimension groups $\big(K_0(\mathrm{B}_1), K_0^+(\mathrm{B}_1), \Gamma(\mathrm{B}_1)\big)$ and $\big(K_0(\mathrm{B}_2), K_0^+(\mathrm{B}_2), \Gamma(\mathrm{B}_2)\big)$ are *isomorphic* if there exists an isomorphism of abelian groups $f : K_0(\mathrm{B}_1) \longrightarrow K_0(\mathrm{B}_2)$ such that $f(K_0^+(\mathrm{B}_1)) = K_0^+(\mathrm{B}_2)$ and $f(\Gamma(\mathrm{B}_1)) = \Gamma(\mathrm{B}_2)$.

Definition

We call a Bratteli diagram B *thick* if for every vertex $v \in V_i$ there exists a vertex $u \in V_k$, for $k > i$, such that v and u are connected by more than one path in B.

Theorem

Let B_1, B_2 *be thick Bratteli diagrams. Then the following conditions are equivalent.*

(i) *The groups* $A(\mathrm{B}_1)$ *and* $A(\mathrm{B}_2)$ *are isomorphic.*

(ii) *The dynamical systems* $(A(\mathrm{B}_1), \mathcal{P}(\mathrm{B}_1))$ *and* $(A(\mathrm{B}_2), \mathcal{P}(\mathrm{B}_2))$ *are topologically conjugate.*

(iii) *The* $*$*-algebras* $\mathcal{M}(\mathrm{B}_1)$ *and* $\mathcal{M}(\mathrm{B}_2)$ *are isomorphic.*

(iv) *The* C^**-algebras* $\mathfrak{A}(\mathrm{B}_1)$ *and* $\mathfrak{A}(\mathrm{B}_2)$ *are isomorphic.*

(v) *The dimension groups* $\Delta(\mathrm{B}_1)$ *and* $\Delta(\mathrm{B}_2)$ *are isomorphic.*

The topological space $\mathcal{P}(\mathrm{B})$ is called *space of paths of the Bratteli diagram* B and that is defined by B. Actually this is also well known constructions introduced by Kerov and Vershik[7]. The group $A(\mathrm{B})$ acts by homeomorphisms on $\mathcal{P}(\mathrm{B})$ faithfully.

[7] A.M. Vershik and S.V. Kerov, *Locally semisimple algebras. Combinatorial theory and the K_0-functor*, Soviet Math.J. **38** (1987), 1701–1733.

In fact, the proof is not so complicated, but it combines into one four powerful technics. Namely,

- construction of space of paths of the Brattely diagram
- the technics of reconstruction of topological spaces from their groups of homeomorphisms
- the technics of crossed products
- classical results on AF-algebras.

Technics of the proof

- The equivalences $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ are classical[8].
- Equivalence $(i) \Leftrightarrow (ii)$ follows from powerful result of M.Rubin[9] on reconstruction of topological spaces from their groups of homeomorphisms.
- Implication $(ii) \Rightarrow (iii)$ obtains due to technics of crossed products.
- Implication $(iv) \Rightarrow (i)$ follows from classical results about AF-algebras.

---

[8] G. A. Elliot, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44.
Bruce Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998.
Kenneth R. Davidson, *C^*-algebras by example*, American Mathematical Society, Providence, RI, 1996.

[9] Matatyahu Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Trans. Amer. Math. Soc. **312** (1989), no. 2, 487–538.

Simple groups

Definition

A Bratteli diagram B is called *simple* if for every infinite path γ in B and vertex $v \in V(\mathrm{B})$ the vertex v is connected by a path starting at v and ending in a vertex on γ.

The condition of simplicity is natural by the following result:

Proposition

The following conditions are equivalent.

1. *The diagram* B *is simple.*
2. *The* C^**-algebra* $\mathfrak{A}(\mathrm{B})$ *is simple.*
3. *The* $*$*-algebra* $\mathcal{M}(\mathrm{B})$ *is simple.*
4. *The dynamical system* $(A(\mathrm{B}), \mathcal{P}(\mathrm{B}))$ *is minimal. (A dynamical system is minimal if every its orbit is dense.)*
5. *The dimension group* $\Delta(\mathrm{B})$ *is simple (has no order ideals).*

Proposition

Let G be a simple LDA-group. Then one and only one of the following cases is possible:

1. *G is isomorphic to Alt_d, where $d \neq 4$.*
2. *G is isomorphic to the infinite alternating group Alt_∞.*
3. *G is isomorphic to $A(\mathsf{B})$, where B is a thick simple Bratteli diagram.*

Example. Diagonal direct limits of finite symmetric and alternating groups

As an example, consider the case of the unital Bratteli diagrams which have only one vertex on each of the levels.

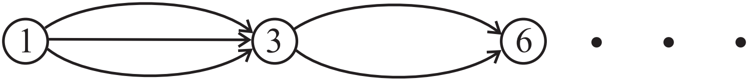

Figure: Bratteli diagram

Proposition

Let B *be a unital Bratteli diagram which has precisely one vertex* v_i *on each of the levels* V_i. *Let* $k_i = |E_i|$ *be the number of the edges of the* i*th level. Denote by* κ *the supernatural number* $\prod_{i=1}^{\infty} k_i$. *Then the dimension group* $\Delta(\mathrm{B}) = \left(K_0(\mathrm{B}), K_0^+(\mathrm{B}), \Gamma(\mathrm{B})\right)$ *is given by*

$$K_0(\mathrm{B}) = \left\{ \frac{p}{q} \ : \ p \in \mathbb{Z}, q \text{ divides } \kappa \right\},$$
$$K_0^+(\mathrm{B}) = K_0(\mathrm{B}) \cap [0, +\infty),$$
$$\Gamma(\mathrm{B}) = K_0(\mathrm{B}) \cap [0, 1].$$

Thus, two such groups are isomorphic if and only if corresponding supernatural numbers are equal. Originally it was proved by V.Sushchansky and N.Kroshko in 1994.

Space of paths

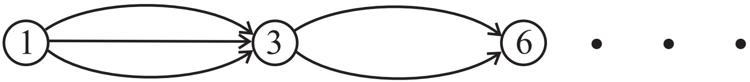

Figure: Bratteli diagram

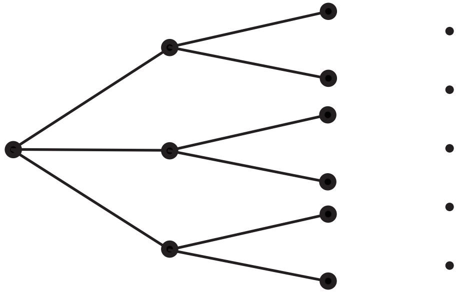

Figure: Spherically homogeneous rooted tree