Groups acting on the spaces of the Bratteli diagram paths

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Introduction

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- Rooted trees, their boundaries, and groups acting on them.
- On classification of inductive limits of direct products of alternating groups.

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Groups which we study in this section:

- The isometry group of ∂T = The isometry group of T = The automorphism group of T
- Homogeneous symmetric group of ∂T
- The locally isometry group of ∂T= Product of the isometry group of ∂T and homogeneous symmetric group of ∂T
- The group of measure-preserving homeomorphisms of $\partial T \supset$ The locally isometry group of ∂T.

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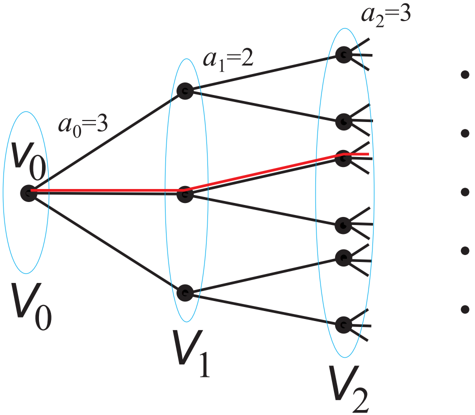
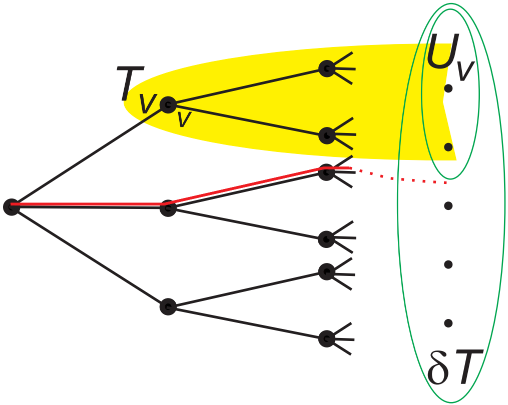


Figure: Spherically homogeneous rooted tree

n-th level is the set $V_n(T) = \{v \in V(T) : d(v_0, v) = n\}$.
Spherical index is the sequence $\Theta = (a_0, a_1, \ldots)$.
Characteristics is the supernatural number $\Omega(\Theta) = \prod_{i=0}^{\infty} a_i$.

Boundary of rooted tree



An end of a rooted tree is an infinite path starting in the rooted vertex and having no repetitions.

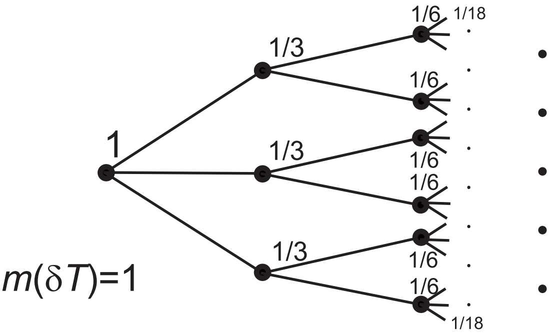
Cylinder set: $U_v = \{x \in \partial T \mid v \in x\}, \quad v \in V(T)$.

Let $\bar{\lambda} = \{\lambda_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers tending to zero. We can introduce a natural ultrametrics on ∂T by putting $\rho(x_1, x_2) = \lambda_n$, where n is the length of the maximal common part of the paths x_1 and x_2.
The topology induced by the metrics ρ (or for convenience $\bar{\lambda}$) is compact, totally disconnected and has a base of open sets (balls or cylinder sets) of the following form

$$U_v = \{x \in \partial T \mid v \in x\}, \quad v \in V(T).$$

This compact ultrametric space will be denoted by $(\partial T, \bar{\lambda})$ or simply by ∂T.

Bernoulli measure (measure on cylinder sets) on boundary of rooted tree



Weakly branch groups

An isometry group is said to be *level-transitive* if it acts transitively on all levels.

Definition

Let $G \leq \mathrm{Isom}\, T$ be an isometry group of the tree T. Then for every vertex $v \in V(T)$ the set of all isometries $g \in G$ fixing all vertices outside the subtree T_v is called *the vertex group* (or *the rigid stabilizer* of the vertex) and is denoted by $\mathrm{rist}_G(v) = \mathrm{rist}(v)$.

Definition

A level-transitive isometry group of a rooted tree is said to be *weakly branch group* if $|\mathrm{rist}(v)| = \infty$ for every $v \in V(T)$.

Rigidity of weakly branch groups

Theorem

Let G_1 and G_2 be weakly branch automorphism groups of spherically homogeneous rooted trees T_1 and T_2 respectively. If $\phi : G_1 \longrightarrow G_2$ is an isomorphism of abstract groups, then there exists a measure-preserving homeomorphism $F : \partial T_1 \longrightarrow \partial T_2$ such that

$$\phi(g)(F(w)) = F(g(w))$$

for all $w \in \partial T_1$ and $g \in G_1$, i.e., such that ϕ is induced by F.

Theorem

Let $G_i \leq \mathrm{Isom}\, T$ be weakly branch groups and let $\phi : G_1 \longrightarrow G_2$ be a saturated isomorphism. Then ϕ is induced by an automorphism F of the rooted tree T.

Corollary

- *The isometry group of spherically homogeneous rooted tree is complete.*
- *The isometry groups of spherically homogeneous rooted trees are isomorphic if and only if corresponding trees are isometric for some metrics.*

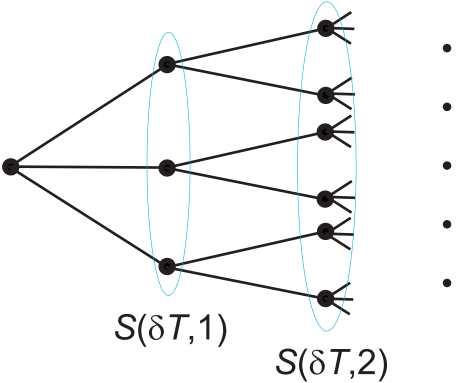
Metrics on the homeomorphism group

We can introduce metrics on the homeomorphism group using metrics ρ (i.e. $\bar{\lambda}$) on the compact space ∂T putting

$$\bar{\rho}(g, h) = \max_{x \in \partial T} \rho(x^g, x^h)$$

for all g and h in $\mathrm{Homeo}\,\partial T$.

Homogeneous symmetric group



$$S(\partial T)=\bigcup_{i=1}^{\infty}S(\partial T,i)$$

The group $S(\partial T)$ is an example of non-finitary locally finite group.

Normal structure of symmetric homogeneous groups

Theorem (Kegel)

1. $S(\partial T_\Theta) = A(\partial T_\Theta)$ *iff* $\Omega(\chi)$ *is divisible by* 2^∞.
2. *If* $\Omega(\Theta)$ *is not divisible by* 2^∞ *then* $[S(\partial T_\Theta) : A(\partial T_\Theta)] = 2$.
3. $A(\partial T_\Theta)$ *is the commutator subgroup* $S(\partial T_\Theta)$.
4. $A(\partial T_\Theta)$ *is a simple group.*

Isomorphisms of homogeneous symmetric groups

Theorem

$S(\partial T_\Theta)$ and $S(\partial T_\chi)$ are isomorphic if and only if

$$char(\Theta) = char(\chi).$$

Automorphisms of homogeneous symmetric groups

Theorem

Every automorphism of the group $S(\partial T)$ is locally inner.
The automorphism group of the subgroup $A(\partial T)$ coincides with the automorphism group of the group $S(\partial T)$.

Local isometries

Definition

A bijection is called local isometry if in some neighborhood of each point it acts as an isometry. More precisely, let (X_1, d_1) and (X_2, d_2) be metric spaces. A bijection

$$\alpha : X_1 \to X_2$$

is called *local isometry* if for every $x \in X_1$ there exists a neighborhood U_x of x such that for every $x_1, x_2 \in U_x$ the following equality holds

$$d_2(x_1^{\alpha}, x_2^{\alpha}) = d_1(x_1, x_2).$$

It is clear that the set of local isometries of a metric space (X, d) forms a group.

Decomposition of the locally isometry group into product of its subgroups

Theorem

The group $\mathrm{LIsom}\,\partial T_{\Theta}$ *is decomposed into product of* $\mathrm{Isom}\partial T_{\Theta}$ *and* $S(\partial T_{\Theta})$.

Normal structure of the locally isometry group

Theorem

Every normal subgroup of the group $\mathrm{LIsom}\,\partial T_{\ominus}$ *contains commutator subgroup* $((\mathrm{LIsom}\,\partial T_{\ominus})' = (\mathrm{Isom}\partial T_{\ominus})' A(\partial T_{\ominus}))$.

Automorphisms of the locally isometry group

Theorem

The group $\operatorname{LIsom} \partial T_{\ominus}$ *is complete.*

Isomorphisms of the local isometry group

Theorem

Let T_1 and T_2 be locally finite rooted trees such that the groups of the local isometries act on their boundaries transitively. The full local isometries groups of $(\partial T_1, \bar{\mu}_1)$ and $(\partial T_2, \bar{\mu}_2)$ are isomorphic if and only if $(\partial T_1, \bar{\lambda}_1)$ and $(\partial T_2, \bar{\lambda}_2)$ are locally isometric for some metrics $\bar{\lambda}_1$ and $\bar{\lambda}_2$. In other words there are positive integers i and j, such that for every natural s the equality holds

$$|V_{i+s}(T_1)| = |V_{j+s}(T_2)|.$$

Groups of measure preserving homeomorphisms

Let $\mathcal{M}$ be the set of all homeomorphisms of ∂T that preserve the Bernoulli measure. It is clear that the set of such homeomorphisms forms a group.

Theorem

The group $\mathcal{M}$ is the closure of $S(\partial T)$ in the topology induced by the metrics $\bar{\rho}$.

$\bar{\rho}(g, h) = \max_{x \in \partial T} \rho(x^g, x^h)$ for all g and h in $\mathrm{Homeo}\, \partial T$.

Automorphisms of the groups of measure preserving homeomorphisms

Theorem

If subgroup G of $\mathcal{M}$ contains weakly branch subgroup, then every automorphism of G is induced by an element of $\mathcal{M}$, that is $Aut(G) \simeq N_{\mathcal{M}}(G)$.

Corollary

The group $\mathcal{M}$ is complete.

Isomorphisms of the groups of measure preserving homeomorphisms

Theorem

Let T_1 and T_2 be spherically homogeneous trees. The following conditions are equivalent:

1. $\mathcal{M}(\partial T_1) \simeq \mathcal{M}(\partial T_2)$;
2. $S(\partial T_1) \simeq S(\partial T_2)$;
3. *The characteristics of the spherical indexes of T_1 and T_2 are equal.*

On classification of inductive limits of direct products of alternating groups

On classification of simple locally finite groups

Classification of locally finite simple groups is very far from being complete, but class of finitary groups were studied in detail.
A *finitary linear representation* of a group G is a linear representation ρ such that for every $g \in G$ the kernel $\ker(1 - \rho(g))$ has finite co-dimension. A group is called *finitary linear* if it has a faithful finitary linear representation.
It is known comlete list of possible finitary linear groups.

The non-finitary locally finite simple groups are understood much worse. There exists a rough division of this class into groups of "1-type, or ∞-type, or p-type". This division uses the notion of a Kegel cover, defined in the following way.

Definition

A set of pairs $\{(H_i, M_i) \mid i \in I\}$ is called a *Kegel cover* for a locally finite group G if, for all $i \in I$, H_i is a finite subgroup of G and M_i is a maximal normal subgroup of H_i, and for each finite subgroup H of G there exists $i \in I$ with $H \leq H_i$ and $H \cap M_i = 1$. The groups H_i/M_i, $i \in I$, are called the *factors* of the Kegel cover.

Every simple locally finite group has a Kegel cover. Using Kegel covers many questions about locally finite simple groups can be transfered to questions about finite simple groups.

Definitions of classes of non-finitary simple locally finite groups

- A non-finitary locally finite simple group is said to be of 1-*type* if every Kegel cover of G has a factor which is an alternating group.
- If p is a prime, then G is of p-*type* if G is non-finitary and every Kegel cover of G has a factor which is isomorphic to a classical group defined over a field of characteristic p.
- A non-finitary locally finite simple group is said to be of ∞-*type* if for any class $\mathcal{G}$ of finite simple groups, such that every finite group can be embedded into a member of $\mathcal{G}$, there exists a Kegel cover of G all of whose factors are isomorphic to a member of $\mathcal{G}$.

U. Meierfrankenfeld and S. Delcroix proved that if G is a locally finite simple group then exactly one of the following possibilities holds:

1. G is finitary;
2. G is of 1-type;
3. G is of p-type for a unique p;
4. G is of ∞-type.

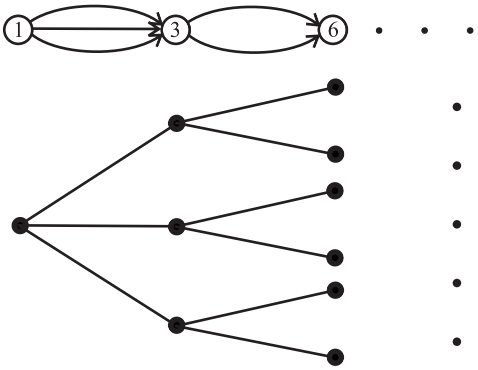
An important class of groups of 1-type are the inductive limits of direct products of alternating groups with respect to block-diagonal embeddings (also called *LDA*-groups). The class of such groups does not exhaust the class of groups of 1-type, but the structure of an arbitrary group of 1-type seems to be similar to the structure of an *LDA*-group.

Definition

LDA-groups are direct limits of direct products $H_i = A_{i1} \times \ldots \times A_{ir_i}$ $(i \in \mathbb{N})$ of finite alternating groups $A_{ik} = \mathrm{Alt}\,(X_{ik})$ such that, for all $i < j$, every non-trivial orbit of any A_{ik} on any X_{jl} is natural.

The LDA-groups can be defined in the terms of Bratteli diagrams.

Example



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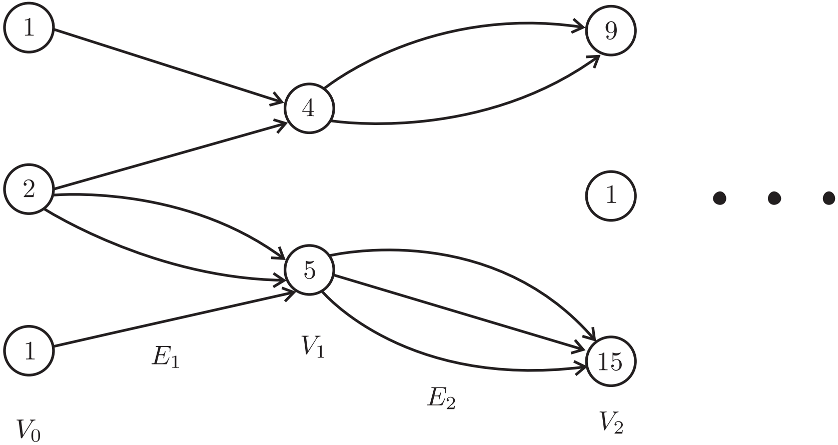

Figure: A Bratteli diagram

We classify locally finite groups, which are inductive limits of direct products of alternating groups with respect to block-diagonal embeddings. This class of groups includes a well known class of simple locally finite groups (so-called LDA-groups). We show that two such groups are isomorphic if and only if the AF-algebras defined by the respective Bratteli diagrams are isomorphic. Then the classical results on classification of AF-algebras can be applied.

In fact, the proof is not so complicated, but it combines into one four powerful technics. Namely,

- construction of space of paths of the Brattely diagram
- the technics of reconstruction of topological spaces from their groups of homeomorphisms
- the technics of crossed products
- classical results on AF-algebras.

- *R. Grigorchuk, V. Nekrashevich, V. Sushchansky* Automata, dynamical systems, and groups. Translation in Proc. Steklov Inst. Math. 2000, no. 4 (231), 128–203
- *Lavrenyuk Ya.* On Automorphisms of Local Isometry Groups of Compact Ultrametric Spaces. Int. J. of Algebra and Computation, 15(5-6):1013–1024, 2005.
- *Ya.Lavreniuk and V.Nekrashevych* Rigidity of branch groups acting on rooted trees. Geometriae Dedicata, 89, N.1-2, (2002), 159-179.
- *Lavrenyuk Ya. and Nekrashevych V.* On classification of inductive limits of direct products of alternating groups. J. of London Math. Soc, 75, (2007), No. 1, p. 146-162.

- *Lavrenyuk Ya., Nekrashevych V., and Sushchansky V.* Diagonal direct limits of symmetric and alternating groups. Submitted to Pacific J. of Math.
- *Ya.V.Lavrenyuk and V.I.Sushchansky* Automorphisms of Homogeneous Symmetric Groups and Hierarchomorphisms of Rooted Trees, Algebra and Discrete Mathematics, N. 4 (2003), 33–49.
- *Ya.V.Lavrenyuk and V.I.Sushchansky* Normal subgroup lattice of the locally isometry group of spherically homogeneous tree boundary. Dopovidi NAN Ukrainy. n.3, 2007, 20–24.
- *V.Nekrashevych* Self-similar groups, volume 117 of Mathematical Surveys and Monographs. Amer. Math. Soc., Providence, RI, 2005, 231p.